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INTERMEDIATE-THRUST ARCS IN MAYER'S VARIATIONAL PROBLEM†

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For Mayer's variational problem of optimum rocket flight in a Newtonian force field, equations (necessary conditions) are written and their analytical solutions, corresponding to motion with intermediate thrust, are obtained. When the flight time is not fixed and the functional of the problem does not explicitly depend on the polar angle, the solutions obtained differ from the known solutions. For flight with a fixed time, the solutions obtained correspond to motion along a certain spiral trajectory. Conditions the satisfaction of which field solutions that satisfy the necessary condition of Robbins optimality are described. The types of problem in which the intermediate-thrust arcs obtained can be used are determined. An example is given. © 2000 Elsevier Science Ltd. All rights reserved.

It is well known that, in Mayer's variational problem of the optimum rocker trajectory in a Newtonian field, satisfaction of the necessary conditions of optimality reduces to integration of a system of fourteenth-order differential equations separately on arcs of zero thrust (ZT), intermediate thrust (IT) and maximum thrust (MT) [1–3]. The IT arcs correspond to a singular solution, and their possible appearance in the problem gives rise to considerable difficulties. There is no general theory for the analysis of these arcs and their optimality [4–6]. Certain analytical solutions are known, such as spiral trajectories [1, 4, 7], spherical trajectories [8]‡ and circular trajectories [8, 9]. However, these solutions provide no answer to the question of their ensuring an effective minimum. Furthermore, certain classes of spiral trajectories have been obtained, differing from previously known solutions and satisfying the necessary condition of Robbins optimality [10]. Questions of the possibility of the existence of optimal IT arcs and their applicability remain open.

In the present paper, on the basis of an analysis of the necessary conditions of optimality–a canonical system of equations of the variational problem and the properties of a switching function–new classes of analytical solutions are obtained for IT arcs. The problems of satisfying the necessary condition of Robbins optimality and the use of the arcs obtained to solve flight dynamics problems are investigated.

1. FORMULATION OF THE PROBLEM

Let the equation of a rocket motion in a central Newtonian force field be given in the form [2]

$$\dot{\mathbf{v}} = cmM^{-1}\mathbf{e} - \mu r^{-2}\mathbf{r}, \quad \dot{\mathbf{r}} = \mathbf{v}, \quad \dot{M} = -m \tag{1.1}$$

where μ is the gravitation parameter, $\mathbf{r}$ is the radius vector drawn from the centre of gravitation, $\mathbf{v}$ is the velocity vector, $\mathbf{e}$ is the unit thrust vector, M is the mass of the rocket, m is the mass consumption per second, with the imposed constraint $0 \leqslant m \leqslant \overline{m}$, and c = const is the gas outflow rate. For the direction cosines e_1, e_2, e_3 of the thrust vector and the mass consumption per second, the following equalities hold [1]

$$e_1^2 + e_2^2 + e_3^2 = 1, \qquad m(\overline{m} - m) - \gamma^2 = 0 \tag{1.2}$$

The control variables are m, e_1, e_2, e_3 and γ. To simplify the notation, the phase variables will be denoted by $x_i (i = 1, \ldots, 7)$, where x_1, x_2 and x_3 are the components of $\mathbf{v}$, x_4, x_5 and x_6 are components of $\mathbf{r}$ and x_7 denotes the mass. At the initial instant $t = t_0$, let the conditions $x_i = x_{i0}$ be specified. When $t = t_1$, the

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‡See also Azimov, D. M., An investigation of the optimum trajectories in a central Newtonian field. Candidate dissertation, Moscow, 1991.

final conditions $x_k = x_{kl}$ are specified, where $k = 1, \dots, j < 7$. It is required to find values of m, e_1, e_2, e_3 and y such that the x_i corresponding to them satisfy equations of motion (1.1), equalities (1.2) for the direction cosines of the thrust vector and mass consumption per second, and the initial and final conditions, and that the prescribed functional

$$J = J(x_{j+1,1},\ x_{j+2,1},\ \dots,\ x_{71},\ t_1)$$

takes the minimum value of those possible.

As is well known from the general theory of the optimal trajectories of rockets in a gravitational field [1], analysis of the Weierstrass conditions leads to conclusions that, along the optimal trajectory, the thrust must be oriented in the direction of the basis vector, and the following conditions must be satisfied: $\chi \leqslant 0$ on ZT arcs, when $m = 0$; $\chi = 0$ on IT arcs, where $0 < m < \bar{m}$; and $\chi \geqslant 0$ on MT arcs, where $m = \bar{m}$. Here, $\chi = cM^{-1}\lambda - \lambda_7$ is the switching function, λ is the magnitude of the basis vector $\boldsymbol{\lambda}$ and λ_7 is a factor conjugate to the mass. At the switching point of two different arcs of the vector function $\boldsymbol{\lambda}$, the vector function $\boldsymbol{\lambda}$ itself and the function λ_7 should be continuous. Taking into account the stationarity and Weierstrass conditions we can write Eqs (1.1) in the form of a closed canonical system of fourteenth-order equations on each thrust section [2]. In the general case, this system can be written in the form

$$\begin{aligned} &\dot{\mathbf{v}} = cmM^{-1}\lambda^{-1}\boldsymbol{\lambda} - \mu r^{-2}\mathbf{r},\quad \dot{\mathbf{r}} = \mathbf{v},\quad \dot{M} = -m \\ &\dot{\boldsymbol{\lambda}} = -\boldsymbol{\lambda}_r,\quad \dot{\boldsymbol{\lambda}}_r = \mu r^{-3}\boldsymbol{\lambda} - 3\mu r^{-5}\,(\boldsymbol{\lambda}\mathbf{r})\mathbf{r},\quad \dot{\lambda}_7 = cmM^{-2}\lambda \end{aligned} \tag{1.3}$$

with the Hamiltonian

$$H = -\mu(\boldsymbol{\lambda}\mathbf{r})\mathbf{r}^{-3} + \boldsymbol{\lambda}_r\mathbf{v} + \chi m$$

In a spherical system of coordinates (r, θ, δ) with its origin at the centre of attraction [2], the vectors have the following components.

$$\begin{aligned} &\mathbf{r} = (r,\ 0,\ 0),\quad \mathbf{v} = (v_1,\ v_2,\ v_3),\quad \boldsymbol{\lambda} = (\lambda_1,\ \lambda_2,\ \lambda_3) \\ &\boldsymbol{\lambda}_r = (\lambda_4,\ [\lambda_1 v_2 - \lambda_2 v_1 + (\lambda_2 v_3 - \lambda_3 v_2)\mathrm{tg}\delta + \lambda_5\cos\delta]r^{-1}\ \ (\lambda_1 v_3 - \lambda_3 v_1 + \lambda_6)r^{-1}) \end{aligned}$$

where $\boldsymbol{\lambda}_r$ is a vector conjugate to the radius vector [2], and λ_4, λ_5, and λ_6 are factors conjugate to r, θ and δ respectively.

2. THE CASE WHEN THE FLIGHT TIME IS FIXED AND THE FUNCTIONAL DEPENDS EXPLICITY ON THE POLAR ANGLE

It can be shown [2, 5] that, in the plane case, when a polar system of coordinates (r, θ) is used with its origin at the centre of attraction, where θ is the angle between the radius vector and the axis $\theta = 0$, the canonical system of equations of motion (1.3) on IT arcs has the following first integrals and invariant relations

$$H = \lambda_1(v_2^2 r^{-1} - \mu r^2) - \lambda_2 v_1 v_2 r^{-1} + \lambda_4 v_1 + \lambda_5 v_2 r^{-1} = C \tag{2.1}$$

$$\lambda_1 v_1 + \lambda_2 v_2 - 2\lambda_4 r + c\lambda\ln(M_0 M^{-1}) - 3Ct = C_1,\quad \lambda_5 = C_3,\quad \lambda_7 M = c\lambda = C_2 \tag{2.2}$$

$$\lambda_1\lambda_4 + \lambda_1\lambda_2 v_2 r^{-1} + \lambda_2^2 v_1 r^{-1} + \lambda_5\lambda_2 r^{-1} = 0 \tag{2.3}$$

$$\lambda_4^2 + (\lambda_1 v_2 r^{-1} - \lambda_2 v_1 r^{-1} + \lambda_5 r^{-1})^2 = \lambda^2\mu r^{-3} - 3\lambda_1^2\mu r^{-3} \tag{2.4}$$

where v_1 and v_2 are the radial and transverse components of the velocity vector, t is the time of motion along the IT arc, M_0 is the initial mass and C, C_1, C_2 and C_3 are integration constants. Eliminating the variable λ_4 and the difference $(\lambda_1 v_2 - \lambda_2 v_1)$ from (2.1), (2.3) and (2.4), and using the last relation of (2.2), we obtain an equation in r

$$\begin{aligned} &C^2\lambda^4 r^4 + 6\mu C\lambda^2\lambda_1^3 r^2 + \mu C_3^2\lambda^4(3\lambda_1^2\lambda^{-2} - 1)r + 9\mu^2\lambda_1^6 = 0 \\ &\lambda_1 = \lambda\sin\varphi,\quad \lambda_2 = \lambda\cos\varphi \end{aligned} \tag{2.5}$$

where λ = const is the basis vector and φ is the angle between the thrust vector and the perpendicular to the radius vector [1]. To solve this equation, we shall examine the following possible cases related to the conditions of the problem.

1. The flight time is not fixed ($C = 0$) and the functional of the problem does not depend explicitly on the polar angle ($C_3 = 0$) [10]. In this case, analysis of relations (2.1), (2.3) and (2.6) taking into account the last equality of (2.2) indicates that IT arcs degenerate into ZT arcs.

2. The flight time is not fixed ($C = 0$) and the functional of the problem depends explicitly on the polar angle ($C_3 \neq 0$). In this case, with $\lambda = 1$, the spiral trajectories obtained by Lawden [1] and Kelley [4] and, for any values of λ, other spiral trajectories which are obtained using the properties of the switching function ($\dot{\chi} = 0$, $\ddot{\chi} = 0$) [10] are known. These solutions differ in the formulae used to determine the variables of the canonical system of equations (1.3), apart from the formula for the radius vector. However, as has been shown [4–6, 10], none of these solutions satisfies the necessary optimality condition.

3. The flight time is fixed ($C \neq 0$) and the functional of the problem does not depend on the polar angle ($C_3 = 0$). For this case, solutions are known in the form of spiral trajectories that differ from the above and satisfy the necessary Robbins optimality condition when $\sin\varphi < 0$ and $C > 0$ [10].

4. The flight time is fixed ($C \neq 0$) and the functional of the problem depends explicitly on the polar angle ($C_3 \neq 0$). This is the most general case in solving Eq. (2.5). Depending on the signs of the discriminant

$$Q = \mu^4 \frac{C_3^4}{C^8}(3s^2-1)^2\left[\frac{C_3^4}{256}(3s^2-1)^2 - \mu C\lambda^3 s^9\right]$$

of Eq. (2.5), where $s = \sin\varphi$, and for expression Cs, we have the conditions

$$\text{(a)}\, Q \geqslant 0,\quad Cs < 0; \qquad \text{(b)}\, Q < 0,\quad Cs > 0$$

Here, the form of the solutions depends on the sign of the expression $3s^2 - 1$. If $3s^2 - 1 < 0$, the corresponding solutions for conditions a and b have been presented earlier [10]. When $3s^2 - 1 = 0$, circular IT arcs occur, the solutions for which were discussed earlier [8, 9]. Other solutions of Eq. (2.5) are given below, revealed as a result of further studies of conditions a and b when $3s^2 - 1 > 0$, taking into account for the signs of the expressions $2\cos(\alpha/3) - 1$ and $1 + 2\operatorname{cosec} 2\alpha$, where α is a known function of the variable s.

We introduce the notation

$$R_i = \left[\frac{\sqrt{2}\mu C_3^2(3s^2-1)}{4\lambda^2 C^2 X_i(s)}\right]^{1/2} - X_i(s), \quad \xi = \frac{C_3^4(3s^2-1)^2}{128\mu\lambda^7 Cs^9} - 1$$

When $Q \geqslant 0$ and $Cs < 0$, we have

$$\begin{aligned} &r_1 = R_1(s),\quad X_1(s) = (-2C^{-1}\mu\lambda s^3(2\operatorname{cosec}2\alpha + 1))^{1/2} \\ &a = \operatorname{arctg}((\operatorname{tg}\beta)^{1/3}),\quad |\alpha| \leqslant \pi/4,\quad 1 + 2\operatorname{cosec}2\alpha > 0,\quad \beta = -1/\xi \end{aligned} \tag{2.6}$$

When $1 + 2\operatorname{cosec} 2\alpha \leqslant 0$, no IT arcs exist; there are no effective solutions.

When $Q < 0$ and $Cs > 0$, we have

$$\begin{aligned} &r_2 = R_2(s),\quad X_2(s) = (2C^{-1}\mu\lambda s^3(2\cos(\alpha/3) - 1))^{1/2} \\ &2\cos(\alpha/3) - 1 > 0,\quad \alpha = \arccos\xi \end{aligned} \tag{2.7}$$

If $2\cos(\alpha/3) - 1 \leqslant 0$, then

$$r_3 = R_3(s),\quad X_3(s) = (2C^{-1}\mu\lambda s^3(1 - 2\cos(\alpha/3 + \pi/3)))^{1/2} \tag{2.8}$$

After determining $r = r(s)$, the remaining solutions of system of equations (1.3), taking into account the relation

$$(\lambda^2 - 5\lambda_1^2)v_1 + 2\lambda_1(\lambda_1 v_1 + \lambda_2 v_2) - 4\lambda_1\lambda_4 r = 0 \tag{2.9}$$

which is obtained from the equality $\dddot{\chi} = 0$, can be written in the form

$$\begin{aligned}
&v_1 = 2k\frac{3w+2C_3}{5s^2-3}, \quad v_2 = \frac{(3-s^2)w+4C_3k^2}{\lambda s(5s^2-3)} \\
&t = \frac{\lambda}{2}\int\frac{(5s^2-3)R}{k(3w+2C_3)}d\varphi + C_4, \quad \theta = \frac{1}{2}\int\frac{(R(3-s^2)w+6k^2\lambda)}{rsk(3w+2C_3)}d\varphi + C_5 \\
&M = M_0 \exp\frac{P}{c\lambda} \\
&\lambda_1 = \lambda s, \quad \lambda_2 = \lambda k, \quad \lambda_4 = k(\lambda - w)/(\lambda s), \quad \lambda_7 = C_2 M^{-1} \\
&w = C_3^{-1}(3\mu\lambda^2 s^4 r^{-1} + C\lambda s r + C_3^2), \qquad R = dr/d\varphi, \quad s = \sin\varphi, \quad k = \cos\varphi \\
&P = C_2 - k\frac{w(15s^2-3)+8s^2C_3}{s(s-3)^2} + 3C(t-C_4)
\end{aligned} \tag{2.10}$$

Here θ is the polar angle, and C_4 and C_5 are integration constants. The mass consumption per second can be determined from the equation $\chi(\mathrm{IV}) = 0$ in the form

$$\begin{aligned}
m = [&10\lambda^2\mu s^2 - v_2^2 r\lambda^2(3-13s^2) - 2\lambda_4 r^2\lambda(sv_1 - 3kv_2) - \\
&-4\lambda_4^2 r^3 + 6\lambda s C_3 v_2 r^{-1}](cs\lambda^2 r^2(3-5s^2))^{-1}
\end{aligned} \tag{2.11}$$

Thus, expressions (2.6)–(2.8), (2.10) and (2.11) are solutions of the canonical system of equations (1.3) irrespective of the optimality criterion, describing motion with IT along certain spiral trajectories. When $s < 0$ and $C > 0$ in formula (2.6), and when $s < 0$ and $C < 0$ in formulae (2.7) and (2.8), the solutions obtained meet the requirements of the necessary optimality condition [5]. These requirements generally [3, 5] reduce to the radial component of the basis vector being negative and the switching function being identically equal to zero.

For the complete determination of the feasibility of the necessary optimality condition of the solutions obtained, following the method described above [5, 6], we check the sign of the reactive acceleration. To do this, we project the first equation of system (1.3) [or (1.1)] onto the direction of the basis vector and obtain [1]

$$\dot{u}_1 - u_2\dot{\psi} = cmM^{-1} - \mu r^{-2}s \tag{2.12}$$

where

$$u_1 = \frac{\lambda_1 v_2 - \lambda_2 v_1}{\lambda}, \quad u_2 = \frac{-\lambda_2 v_1 + \lambda_1 v_2}{\lambda}, \quad \psi = \frac{\pi}{2} - \varphi + \vartheta \tag{2.13}$$

where u_1 and u_2 are the projections of the rocket velocity onto the direction of the basis vector and perpendicular to it, and ψ is the angle between the basis vector and the polar axis ($\theta = 0$). In the solutions obtained above, the angle φ is regarded as an independent variable, and therefore we shall write the derivatives of u_1, θ and λ_1 in the form [2, 9]

$$\dot{u}_1 = \frac{du_1}{d\varphi}\frac{d\varphi}{dt}, \quad \dot{\theta} = \frac{v_2}{r}, \quad \dot{\lambda}_1 = \lambda_2, \quad \dot{\varphi} = \lambda_2\frac{v_2}{r} - \lambda_4 \tag{2.14}$$

Taking account of the relations $r = r(s)$, (2.10), (2.13) and (2.14), and assuming that $\lambda = 1$ [1, 11], from (2.12), after reduction, we obtain the following expression for the reactive acceleration

$$a = \frac{cm}{M} = \frac{1}{k}\left[\frac{du_1}{d\varphi}\left(\frac{v_2}{r} - \lambda_4\right) - \lambda_4(sv_2 - kv_1) + \mu\frac{s}{r^2}\right]$$

where

$$\frac{du_1}{d\varphi}=-\left(a_3w+a_4C_3+a_5sk\frac{dw}{d\varphi}\right)(s^2(3-5s^2))^{-2}$$

$$\frac{dw}{d\varphi}=-\left(a_1r+a_2\frac{dr}{d\varphi}\right)(r^2C_3)^{-1},\quad \frac{dr}{d\varphi}=v_1\left(\frac{v_1}{r}-\frac{\lambda_4}{k}\right)$$

$$a_1=12\mu s^3k+Cr^2k+2Csr\frac{dr}{d\varphi}+C_3^2\frac{dr}{d\varphi}$$

$$a_2=3\mu s^4+Csr^4+C_3^2r,\quad a_3=-24+40s^4+114s^2k^2+3k^4-15s^2k^4$$

$$a_4=-12+20s^2+40s^2k^2,\quad a_5=24-40s^2-3k^2+5s^2k^2$$

Calculations carried out for various values of constants C and C_3 showed that, for the set of s values satisfying the inequalities $s = \sin\varphi < 0$ and $(3s^2 - 1) > 0$, the following conditions hold

$$a>0,\ \text{if}\ \ -1<s<-0{,}7746;\ a<0,\ \text{if}\ \ -0{,}7746\leqslant s<0$$

Consequently, the classes of solutions obtained above satisfy the necessary optimality condition if $-1 < s < -0.7746$.

3. THE CASE OF FLIGHT WITH A NON-FIXED TIME

If, according to the conditions of the variational problem, the flight time is not fixed, then we have $C = 0$ [1]. Then, from (2.5) we obtain

$$r=9\mu\lambda^2C_3^{-2}s^6(1-3s^2)^{-1} \tag{3.1}$$

Note that this formula was obtained by Lawden [1] and Kelley [4] (for the case $\lambda = 1$).

Below, we shall find analytical solutions corresponding to (3.1) that differ from the solutions presented earlier [1, 4, 10]. Thus, from Eqs (2.3)–(2.5), taking (2.9) into account, we have

$$\lambda(sv_2-kv_1)+C_1=C_3\frac{1-3s^2}{3s^2} \tag{3.2}$$

$$\lambda_4=C_3^3k\frac{(1-3s^2)^2}{27\lambda^2\mu s^9} \tag{3.3}$$

$$\lambda(sv_1+kv_2)=4C_3k\frac{1-3s^2}{6s^3}+\lambda v_1\frac{5s^2-1}{2s} \tag{3.4}$$

Relations (3.2) and (3.4) enable us to find the velocity components

$$v_1=\frac{2C_3k}{3\lambda s^2(3-5s^2)},\quad v_2=\frac{5C_3(1-5s^2+6s^4)}{\lambda s^3(3-5s^2)} \tag{3.5}$$

The remaining solutions, corresponding to (3.1), (3.3) and (3.5), can be obtained in quadratures

$$\begin{aligned} t&=\frac{81\mu\lambda^3}{C_3^3}\int\frac{3-11s^2-10s^4}{1-3s^2}d\varphi+t_0\\ \theta&=3\lambda\int\frac{(3-23s^2-30s^4)(1-2s^2)}{s(1-3s^2)}d\varphi+\theta_0\end{aligned} \tag{3.6}$$

where t_0 and θ_0 are integration constants. Consequently, the first integrals, defined by the first and last relations of (2.2), enable us to determine the remaining variables

$$M = M_0 e^{f(s)},\ \lambda_7 = C_2 M_0^{-1}\exp(f(s))$$
$$f(s) = \left[\frac{C_3\lambda k(5s^2-1)}{3s^2(3-5s^2)} - C\right]C_2^{-1} \tag{3.7}$$

The mass consumption per second can be determined from the equation $\chi^{(IV)} = 0$. Thus, equalities (3.1), (3.3) and (3.5)–(3.7) are solutions of system (1.3) for IT arcs in the case of flight with a non-fixed time as a function of the angle φ. These arcs are spiral trajectories differing from Lawden spirals [1] and other previously known solutions [4, 7, 10]. The solutions given in these papers describe motions along different spiral trajectories, where the polar angles have different rates of change, and the directions of these motions have dissimilar angles of inclination to the horizontal. Furthermore, the actual mass of the rocket is determined by means of various formulae.

To check the necessary condition of Robbins optimality, we shall determine the sign of the reactive acceleration. Taking account of relations (2.13), (2.14), (3.1), (3.3) and (3.5), and assuming that $\lambda = 1$, from (2.12) we obtain

$$a = \frac{cm}{M} = C_3^2 \frac{1-3s^2}{[81\mu\lambda_4 s^{13}(3-5s^3)]}\times$$
$$\times\{-90+943s^2-3768s^4+7620s^6-9430s^8+8625s^{10}-4500s^{12}\} \tag{3.8}$$

Calculations showed that, in the set of s values satisfying the inequalities $s = \sin\varphi < 0$ and $1 - 3s^2 > 0$ (see formula (3.1)), we have $a > 0$ if $-0.5052 < s < 0$, and $a < 0$ if $-0.5773 \leqslant s \leqslant -0.5052$.

Therefore, the class of solutions (3.1), (3.3), (3.5) obtained above satisfies the necessary optimality criterion ($s < 0$) [5] provided $-0.5052 < s < 0$, and consequently is extremal, in unlike the Lawden spirals and other solutions obtained earlier [10].

Remark 1. Earlier [1, 4], in obtaining solutions for IT arcs, no account was taken of the necessary condition of existence of these arcs, which is expressed by the identity $\chi \equiv 0$ [12]. Corresponding analysis of the Lawden solutions showed that they do not satisfy the equalities $\ddot{\chi} = \dddot{\chi} = \chi^{(IV)} = 0$. It is allowance for these conditions that leads to degeneration of the IT arcs found by Lawden [1].

Remark 2. Lawden [11] obtained solutions for a spiral IT are in the case of a non-fixed time and minimization of mass consumption, on the assumption that the trajectory of such an arc occurs in the solution of the problem of optimal take-off from a circular orbit. However, in the given case, taking into account the last equality of (2.2), from the condition of transversality it follows that $C_3 = -\partial J/\partial\theta_1 = 0$. Then, using the fact that $\lambda_5 = -\lambda_A$ [10], where A is a constant occurring in the Lawden solutions, we obtain $A = 0$, which leads to degeneration of the given sections. Consequently, the IT arc obtained by Lawden cannot be included in the optimal take-off trajectory from a circular orbit.

4. THE CASE WHERE THE FUNCTIONAL DOES NOT DEPEND EXPLICITLY ON THE ANGULAR RANGE OF FLIGHT

In the given case, from the condition of transversality, for the final instant of time we obtain [1]

$$\lambda_{51} = C_3 = -\partial J/\partial\theta_1 = 0 \tag{4.1}$$

Then, for the case of a fixed instant of time ($C \neq 0$), the equality

$$r = (-3\mu\, C^{-1}\sin^3\varphi)^{1/2} \tag{4.2}$$

follows from Eq. (2.5), with $C < 0$, $\sin\varphi > 0$ or $C > 0$, $\sin\varphi < 0$.

Further, the following solutions for the arcs examined can be obtained from relations (2.3), (2.4) and (2.9)

$$v_1 = \frac{2}{3}\frac{kz(\varphi)}{\lambda(3-5s^2)},\quad v_2 = \frac{2}{3}\frac{(5-7s^2)z(\varphi)}{\lambda s(3-5s^2)} \tag{4.3}$$

$$\theta = -\frac{15}{4}\operatorname{ctg}\varphi - \frac{21}{4}\varphi + \theta_0, \quad t = -\frac{9}{4}\int \frac{(3-5s^2)}{[r\lambda^2\mu(1-3s^2)]^{1/2}} d\varphi + t_0$$

$$z(\varphi) = (Cr\lambda s + \mu\lambda^2 s^2 r^{-1})^{1/2}$$

The first integrals of (2.1) and (2.2) enable us to obtain solutions for the mass and for the variable conjugate to it in the form

$$M = M_0 \exp(w(\varphi)), \quad \lambda_7 = C_2 M_0^{-1} \exp(-w(\varphi)) \tag{4.4}$$

$$w(\varphi) = \left[\frac{k(5s^2-1)}{s(3-5s^2)} - 3C(t-t_0) - C_1\right](c\lambda)^{-1}$$

The mass consumption per second can now be determined by means of a time derivative of the mass. Consequently, system (4.2)–(4.4) is a class of analytical solutions in quadratures for spiral IT arcs in the case where the minimized functional of the problem does not explicitly depend on the polar angle. Note that another class of IT arcs was obtained earlier [10]. Although in the given solutions the formulae for the radius vector are identical, the behaviour of the other variables is described by different expressions. Motions along the spirals indicated occur at different velocities, rates of change of the polar angle and changes in mass and time. If the direction of motion described by the solutions obtained earlier [10] makes with the perpendicular to the radius vector an angle equal to 3 φ, then the corresponding angle for the solutions obtained above will be equal to $\sim 0.4\ \varphi$. When $C > 0$ and $\sin\varphi < 0$, solutions (4.2)–(4.4) meet the requirements of the necessary optimality condition.

We shall check the sign of the reactive acceleration a. Using (2.13), (2.14) and (4.1)–(4.3), from (2.12) we obtain

$$a = \frac{cm}{M} = \frac{du_1}{d\varphi}\frac{d\varphi}{dt} - u_2\dot{\psi} + \frac{\mu}{r}s \tag{4.5}$$

where

$$\frac{du_1}{d\varphi}\frac{d\varphi}{dt} = \frac{5(1-s^2)s^2}{3-5s^2}\left[(3-5s^2)sk\frac{dz}{d\varphi} - 3z(1-3s^2)\right]$$

$$\frac{dz}{d\varphi} = \mu\lambda^2\left[2sk(1-6s^2)r - s^2)(1-3s^2)\frac{dr}{d\varphi}\right]$$

$$\frac{dr}{d\varphi} = -\frac{9\mu\lambda k}{2C}s^2 r^{-1}$$

From the numerical calculations carried out using formula (4.5) for different values of C ($C > 0$) and the variable s ($s < 0$, $1-3s^2 > 0$), it follows that

for $10^{-6} \leqslant C < 1.1 \times 10^{-5}$

$a < 0$ when $-0.1301 < s < 0, -0.5773 < s < -0.4101$

$a > 0$ when $-0.4101 \leqslant s \leqslant -0.1301$

for $1.1 \times 10^{-5} \leqslant C < 10^{-4}$

$a < 0$ when $-0.5773 < s \leqslant -0.4901$

$a > 0$ when $-0.4901 < s < 0$

for $10^{-4} \leqslant C \leqslant 4 \times 10^{-3}$

$a < 0$ when $-0.5773 < s \leqslant -0.5100$

$a > 0$ when $-0.5100 < s < 0$

and for $C > 4 \times 10^{-3}$ for any s we obtain $a < 0$. Consequently, the solutions obtained in this section satisfy the Robbins condition only for the s values determined above for which the reactive acceleration is positive.

5. EXAMPLE

Using of the results obtained above, we shall examine the problem of determining the flight trajectory for a fixed time between prescribed coaxial elliptical orbits with eccentricities e_1 and e_2 and parameters p_1 and p_2 in the central force field with IT. We shall assume zero perigee longitudes of the orbits. The characteristic velocity will be adopted as the minimized functional. In this case, $\lambda(t_1) = 1$, where t_1 is the final instant of flight. Below we shall show that, for certain values of e_1 and e_2, this flight can be carried out using a single IT arc. In this case, the extremal can consist of a sequence of ZT, IT and ZT, where the ZT arcs are arcs of limiting orbits. Consequently, the flight trajectory will contain two transfer points.

To determine the flight trajectory using the solutions obtained in Section 4, we shall examine the conditions of continuity of the radius vector and velocity vector at these points

$$r_i^2 = -\frac{3\mu}{C}s_i^2 = \frac{p_i^2}{(1+e_i\cos f_i)^2} \tag{5.1}$$
$$\frac{2c_iz_i}{3-5s_i^2} = \left(\frac{\mu}{p_i}\right)^{1/2} e_i\sin f_i, \quad \frac{(5-7s_i^2)z_i}{s_i(3-5s_i^2)} = \left(\frac{\mu}{p_i}\right)^{1/2}(1+e_i\cos f_i)$$
$$c_i = \cos\varphi_i, \quad s_i = \sin\varphi_i, \quad z_i = (Cr_is_i + s_i^2\mu r_i^{-1})^{1/2}, \quad i = 1,2$$

where f_i is the true anomaly. The subscript i denotes values of the corresponding variables at the first and second transfer points.

From relation (5.1) we obtain the values φ_i, f_i and C, where s_i^2 are the solutions of the equations

$$\begin{aligned}&(1+(1-b_i^2/4)/2)[-b_i^2(e_i^2+1)+(b_i^2e_i/2)^2-\\&-2b_i(4(e_i^2+1)-b_i^2e_i^2)^{1/2}+4]^{1/2} = a_i^2(1-3s_i^2)\end{aligned} \tag{5.2}$$

where

$$a_i = \frac{5-7s_i^2}{3-5s_i^2}, \quad b_i = \frac{5-7s_i^2}{s_ic_i}$$

The remaining unknown quantities f_i and C can be determined as follows:

$$\begin{aligned}&f_i = \arcsin\frac{-b_i+(4(e_i^2+1)-b_i^2e_i^2)^{1/2}}{2b_i(1-b_i^2/4)}\\&C = -3\mu s_1^3(1+e_1\cos f_1)^2p_1^{-1}\end{aligned} \tag{5.3}$$

An investigation of relations (5.1)–(5.3) showed that the values of φ_i and f_i depend only on e_i and do not depend on the parameters of the limiting orbits, but they should be related by the equality

$$\frac{s_1^3}{s_2^3}\frac{(1+e_1\cos f_1)^2}{(1+e_2\cos f_2)^2} = \frac{p_1^2}{p_2^2} \tag{5.4}$$

To determine the basis vector on the limiting orbits, we shall examine its continuity conditions at the transfer points [1]

$$\begin{aligned}&\sin\varphi_i = B_ie_i\sin f_i + CI_{2i}(f_i, e_i)\\&\cos\varphi_i = B_i(1+e_i\cos f_i) + \frac{D_i}{1+e_i\cos f_i} + CI_{2i}(f_i, e_i)\\&I_{2i} = \frac{\operatorname{ctg} f_i}{e_i(1+e_i\cos f_i)} + \frac{1+e_i\cos f_i}{e_i\sin f_i}I_{1i}\\&I_{1i} = \sin f_i\int\frac{df}{\sin^2 f(1+e\cos f)^2}\end{aligned} \tag{5.5}$$

where B_i and D_i are unknown constants. From equalities (5.5), B_i and D_i are determined, and the basis vectors on the limiting orbits are thereby determined.

To illustrate the results obtained, we considered a numerical example with specific initial and final conditions: $e_1 = 0.21, e_2 = 0.27, p_1 = 9000$ km, $p_2 = 9732$ km, $\omega_1 = \omega_2 = 0$. From relations (5.1)–(5.4), for the transfer points we obtain

$$r_1 = 7628 \text{ km}, \sin\varphi_1 = -0.4872, \sin f_1 = 0.5166$$

$$r_1 = 7818 \text{ km}, \sin\varphi_2 = -0.4953, \sin f_2 = 0.4225$$

The Hamiltonian constant $C = 0.002377$. The time of motion along the IT arc amounts to 109.8 s and here the ratio of the characteristic velocity ΔV_1 to the local angular velocity V_0 for the given flight is 0.6946.

To compare the results obtained with the results of the solution of similar problems, calculations for a flight were carried out using two arcs of maximum thrust (MT) and for double-impulse flight between prescribed orbits. For MT arcs satisfying the necessary optimality conditions, we used approximate analytical solutions in the case of a linear central field (where the gravitational acceleration is a linear function of the radius vector [3]). Here, the first transfer point lies on the initial elliptical orbit, the second and third points lie on the transitional Kepler orbit and the fourth transfer point connects the transitional orbit to the final elliptical orbit. It turned out that, for flight using MT arcs, the dimensionless quantity characterizing the mass consumption $\Delta V_2/V_0 \cong 0.3008$, and the losses of gravitation acceleration at the second and third transfer points are 0.1×10^{-6} km/s^2 and 1.2×10^{-6} km/s^2 respectively. For Hohmann double-impulse flight we have $\Delta V/V_0 = 0.2859$. Consequently, to achieve the flight in question between the prescribed orbits, the effectiveness (in the sense of fuel consumption) of using IT is 2.3 times lower than in the case of using MT arcs, and 2.4 times lower than when using impulse thrust.

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